Mirror symmetry

Let (M, ω) be a symplectic manifold, and $J \in \mathrm{End}(TM)$ an almost-complex structure compatible with ω.

Defn: If (Σ, j) is a Riemann surface (where $j: T\Sigma \to T\Sigma$, $j^2 = -\mathrm{Id}$, is the complex structure), then a map

$$u: \Sigma \to M$$

is J-holomorphic if

$$Du \circ j = J \circ Du.$$

If $\varphi: \Sigma \to \Sigma$ is biholomorphic, then $u \circ \varphi$ is also J-holomorphic; we say u and $u \circ \varphi$ are related by holomorphic reparametrization.

Today we'll only consider $(\Sigma, j) = \mathbb{CP}^1$, so $\varphi: \Sigma \to \Sigma$ biholomorphic $\Leftrightarrow$ φ is a Möbius transformation

$$\varphi(z) = [az + b : cz + d].$$

If $\beta \in H_2(M; \mathbb{Z})$, we define

$$\mathcal{M}_{0,\beta}(M) := \{u : \mathbb{CP}^1 \to M \text{ J-holomorphic}, [u] = \beta\}/\sim$$

where $u \sim u' \Leftrightarrow u' = u \circ \varphi$ for some φ.

More generally,

$$\mathcal{M}_{k,\beta}(M) := \{ u : \mathbb{CP}^1 \to M \text{ J-holomorphic, } [u] = \beta, \text{ together with a choice of } k \text{ points } z_1, \dots, z_k \in \mathbb{CP}^1 \} / \sim$$

where $u \sim u' \iff u' = u \circ \varphi$ for some φ, with $\varphi(z_i') = z_i$ for all i.

Given smooth cycles $C_1, \dots, C_k \subset M$, we define

$$\mathcal{M}_{k,\beta}(M; C_1, \dots, C_k) := \{ u \in \mathcal{M}_{k,\beta}(M) : u(z_i) \in C_i \text{ for all } i \}.$$

⚠

Under 'certain hypotheses', these moduli spaces are manifolds of dimension

$$2(n-3) + 2c_1(\beta) + \sum_{i=1}^{k} (2 + \dim C_i - 2n)$$

$\dim_{\mathbb{R}}(M) = 2n$ $\qquad$ $c_1 = c_1(TM, J)$.

If this dimension is 0, we can define

$$\langle \alpha_1, \dots, \alpha_k \rangle_\beta := \# \mathcal{M}_{k,\beta}(M; C_1, \dots, C_k)$$

where $\alpha_i \in H^*(M)$ are cohomology classes Poincaré dual to C_i.

These are called <u>Gromov-Witten invariants.</u>

It's convenient to write

$$\langle \alpha_1, \dots, \alpha_k \rangle := \sum_\beta \langle \alpha_1, \dots, \alpha_k \rangle_\beta \, t^{\omega(\beta)}$$

Thm: If $u: \mathbb{CP}^1 \to M$ is a J-hol. curve, then
$$\omega(u) \geq 0,$$
and $\omega(u) = 0 \Leftrightarrow u$ is constant.

Pf: $$\omega(u) = \int_{\mathbb{CP}^1} \omega\left(\frac{\partial u}{\partial s}, \frac{\partial u}{\partial t}\right) ds\, dt$$

$$= \int_{\mathbb{CP}^1} \omega\left(\frac{\partial u}{\partial s}, J\frac{\partial u}{\partial s}\right) ds\, dt$$

(if $s+it$ local holom. coord)

$$= \int_{\mathbb{CP}^1} \left\|\frac{\partial u}{\partial s}\right\|^2 ds\, dt \qquad \omega(\cdot, J\cdot) = \text{metric}$$

$$\geq 0$$

with equality iff $\frac{\partial u}{\partial s} = \frac{\partial u}{\partial t} = 0$ everywhere, i.e., u constant.

In particular, if we set $t=0$,

$$\langle \alpha_1, \dots, \alpha_k \rangle\Big|_{t=0} = \langle \alpha_1, \dots, \alpha_k \rangle_{\beta=0}$$

$$= \#\big(\text{const. maps } u: \mathbb{CP}^1 \to M,\ u(z_i) \in C_i = \text{P.D.}(\alpha_i)\big)$$

$$= C_1 \cap \dots \cap C_k$$

$$= \int_M \alpha_1 \cup \dots \cup \alpha_k$$

where $\int_M : H^{2n}(M) \to \mathbb{C}$ denotes pairing with fundamental class $[M] \in H_{2n}(M)$.

E.g. $M = \mathbb{CP}^n$, $\alpha = $ P.D. (pt)

$H_2(M) \cong \mathbb{Z}$

$\langle \alpha, \alpha \rangle_1$ = # lines through 2 points in $\mathbb{CP}^n$

= 1.

E.g. M = smooth cubic surface in $\mathbb{CP}^3$

$\langle\ \rangle$ = #(lines on a cubic surface) $t^{\omega(\text{line})}$

no point constraints $u(z_i) \in C_i$

= $27\,t$

(where we choose ω so that $[\omega] = c_1(\mathcal{O}(1))$).

E.g. M = smooth quintic hypersurface in $\mathbb{CP}^4$. ("quintic 3-fold")

$\langle\ \rangle$ = #(lines on quintic) t

+ #(conics on quintic) t^2

+ #(cubics on quintic) t^3

+ ...

$= 2875\,t + 609250\,t^2 + 317206375\,t^3 + \dots$

In 1990, only first 3 terms were known. Enter the string theorists!

Defn: We equip $H^*(M)$ with a new family of products, called quantum cup product, parametrized by t:

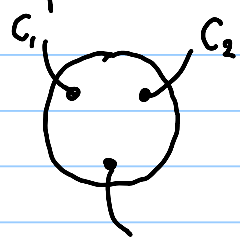
$$*_t : H^*(M) \otimes H^*(M) \to H^*(M)$$

$$\langle \alpha *_t \beta, \gamma \rangle := \langle \alpha, \beta, \gamma \rangle$$

Poincaré duality pairing

Gromov-Witten invariant.

Another way to think about it: You can think about $H^*(M)$ in terms of the Poincaré dual cycles; then cup product corresponds to intersection of cycles. Quantum cup product is:



moduli space of J-hol. spheres with 2 points constrained to lie on cycles C_1, C_2; the third point 'sweeps out' $C_1 *_t C_2$ (weighted by $t^{\omega(\beta)}$, β = homology class).

Note: if we set $t = 0$, then we only count constant spheres, so we just intersect C_1 and C_2; so

$$*_{t=0} = \text{usual cup product.}$$

We say the quantum cup product is a <u>deformation</u> of the usual cup product. We denote $QH^*(M) := (H^*(M), *_t)$, <u>quantum cohomology</u> of M.

It is not just an algebra; it is a Frobenius algebra:

$$\langle \alpha * \beta, \gamma \rangle = \langle \alpha, \beta * \gamma \rangle = \langle \alpha, \beta, \gamma \rangle.$$

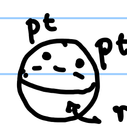
<u>Theorem</u>: • $*_t$ is associative

• $1 *_t \alpha = \alpha$ for <u>all</u> t.

E.g. $M = \mathbb{CP}^1$. $QH^*(M) \cong \mathbb{C}1 \oplus \mathbb{C}\theta$

$\uparrow$ P.D.(pt)

$\theta *_t \theta = t \cdot 1$ counts



remaining pt sweeps out $[\mathbb{CP}^1]$.

$$QH^*(M)_t \cong \mathbb{C}[\theta]/\theta^2 = t.$$

Physicists call $QH^*(M)$ the <u>A-model</u> of M. The A-model is always a <u>symplectic</u> invariant.

There is another model for string theory on a Kähler manifold M, which is called the <u>B-model</u>. The B-model is always a <u>complex</u>/<u>algebraic geometry</u> invariant.

$HT^*(M) := H^*(M, \Lambda^* TM)$ (sheaf cohomology)

$$\cong H^*\Big(\bigoplus_{\ell, m} \Omega^{0,\ell}(\Lambda^m TM), \ \bar{\partial}\Big)$$

(Dolbeault theorem)

global sections of sheaf whose sections locally look like

$$f \cdot d\bar{z}_1 \wedge \dots \wedge d\bar{z}_\ell \wedge \frac{\partial}{\partial z_{\ell+1}} \wedge \dots \wedge \frac{\partial}{\partial z_{\ell+m}},$$

with

$$\bar{\partial}(f \, d\bar{z}_1 \wedge \dots) = \bar{\partial} f \wedge d\bar{z}_1 \wedge \dots$$

This has a natural algebra structure, given by wedging sections. If M is equipped with a non-vanishing $(n,0)$-form Ω, then we can define a Frobenius algebra structure, by

$$\langle \alpha, \beta \rangle := \int_M \iota_\alpha \Omega \wedge \iota_\beta \Omega$$

contracting polyvectors with $(n,0)$ form Ω.

If we have a family of varieties (M_t, Ω_t), we obtain a family of Frobenius algebras parametrized by t, like we got from quantum cohomology.

Mirror Symmetry Conjecture:

There are pairs of Kähler manifolds, M and W (called 'mirror pairs'), so that

$$\begin{tikzcd} \text{A-model}(M) \arrow[dr, leftrightarrow, "\cong"] & \text{A-model}(W) \arrow[dl, leftrightarrow, "\cong"'] \\ \text{B-model}(M) & \text{B-model}(W) \end{tikzcd}$$

In 1991, string theorists Candelas, de la Ossa, Greene, Parkes cooked up the mirror W_t to the quintic 3-fold M. They predicted

A-model

$$\langle \omega *_t \omega *_t \omega, 1 \rangle = \sum_{d \geq 1} \# \begin{pmatrix} \text{deg-}d \text{ curves} \\ \text{on } M \end{pmatrix} t^d$$

B-model

$$\langle \kappa \wedge \kappa \wedge \kappa, 1 \rangle = \int_W \iota_{\kappa \wedge \kappa \wedge \kappa} \Omega_t \wedge \Omega_t$$

$$= 2875t + \dots$$

hence predicted # degree-d curves for <u>all</u> d! And they were right!

Here κ = Kodaira-Spencer class of family W_t, which lies in

$$H^1(TW_t)$$

$\Rightarrow$ $\iota_{\kappa \wedge \kappa \wedge \kappa} \Omega$ is a $(0,3)$-form

$\Rightarrow$ $\iota_{\kappa \wedge \kappa \wedge \kappa} \Omega \wedge \Omega$ is a $(3,3)$-form, so can be integrated. x is mirror to $\omega \in H^2(M)$.

Their predictions have been proven for large class of predicted mirrors M, W, by computing both sides. But

- don't know for higher genus
- don't know full extent of MS
- Kontsevich's HMS categorifies, gives connections with gauge theory, symplectic topology ...